Variations on an algebraic inequality

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Abstract

We present some techniques used to prove a variety of algebraic and geometric inequalities.

1 Main Theorem

Let

$$\Delta(x,y,z) = 2xy + 2yz + 2zx - x^2 - y^2 - z^2.$$

Lemma 1. *Let α, β, γ be positive real numbers such that $\Delta(\alpha,\beta,\gamma) > 0$. Then*

$$\alpha vw + \beta uw + \gamma uv \le 0 \qquad (A)$$

for all real numbers u, v, w with $u + v + w = 0$.

In addition, $\alpha vw + \beta uw + \gamma uv = 0$ if and only if $u = v = w = 0$.

Proof. Indeed,

$$\begin{aligned}\alpha vw - \beta wu - \gamma uv = -\gamma uv + (u+v)(\alpha v + \beta u) &= -\left[(\alpha+\beta-\gamma)uv + \alpha v^2 + \beta u^2\right] \\ = \alpha\left[v + \frac{(\alpha+\beta-\gamma)u}{2\alpha}\right]^2 &+ \frac{u^2\left(2\alpha\beta + 2\beta\gamma + 2\gamma\alpha - \alpha^2 - \beta^2 - \gamma^2\right)}{4\alpha} \\ \ge 0.\end{aligned}$$

It easily follows that equality occurs if and only if $u = v = w = 0$.

Corollary. Let α, β, γ be positive real numbers such that $\alpha + \beta + \gamma = 1$ and $\Delta(\alpha,\beta,\gamma) > 0$. If x, y, z are real numbers with $x + y + z = 1$, then

$$2\sum_{cyc} x\beta\gamma - \sum_{cyc} \alpha yz \ge 3\alpha\beta\gamma. \qquad (B)$$

Proof. Let $u = x - \alpha,\ v = y - \beta,\ w = z - \gamma$. Then $x = u + \alpha$, $y = v + \beta,\ z = w + \gamma$, $u + v + w = 0$, and

$$\begin{aligned}2\sum_{cyc} x\beta\gamma - \sum_{cyclic} \alpha yz &= 2\sum_{cyclic}(u+\alpha)\beta\gamma - \sum_{cyclic}\alpha(v+\beta)(w+\gamma) \\ &= 6\alpha\beta\gamma + 2\sum_{cyc} u\beta\gamma - 3\alpha\beta\gamma - \sum_{cyclic}\alpha vw - \sum_{cyclic}(\alpha\gamma v + \alpha\beta w) \\ &= 3\alpha\beta\gamma - \sum_{cyclic}\alpha vw.\end{aligned}$$

From (A) we have $\sum_{cyclic} \alpha vw \leq 0$, hence (B) follows. Equality occurs only when $x = \alpha,\ y = \beta,\ z = \gamma$.

Theorem 1.

i. Let α, β, γ be positive real numbers such that $\Delta(\alpha, \beta, \gamma) > 0$ and let x, y, z be any real numbers. Then

$$\alpha yz + \beta zx + \gamma xy \leq \frac{\alpha\beta\gamma\,(x+y+z)^2}{\Delta(\alpha,\beta,\gamma)} \qquad (C)$$

with equality if and only if

$$x : y : z = (\beta+\gamma-\alpha)\, a : \beta(\gamma+\alpha-\beta) : \gamma(\alpha+\beta-\gamma)\,.$$

ii. Inequality (C) is a particular case of inequality (A).

Proof. **i.** If $x+y+z=0$, then (C) holds because $\alpha yz + \beta zx + \gamma xy \leq 0$, by (A). If $x+y+z \neq 0$, we can assume that $x+y+z=1$. Let $\Delta = \Delta(\alpha,\beta,\gamma)$ and let

$$u = x - \frac{\alpha(\beta+\gamma-\alpha)}{\Delta},\ v = y - \frac{\beta(\gamma+\alpha-\beta)}{\Delta}\ , w = z - \frac{\gamma(\alpha+\beta-\gamma)}{\Delta}.$$

Because $\sum_{cyc} \frac{\alpha(\beta+\gamma-\alpha)}{\Delta} = 1$ and $x+y+z=1$, we have $u+v+w=0$ and

$$\begin{aligned}
\sum_{cyc} cxy &= \sum_{cyc} c\left(u + \frac{\alpha(\beta+\gamma-\alpha)}{\Delta}\right)\left(v + \frac{\beta(\gamma+\alpha-\beta)}{\Delta}\right) \\
&= \sum_{cyc} cuv + \sum_{cyc} \frac{\alpha\beta\gamma(\beta+\gamma-\alpha)(\gamma+\alpha-\beta)}{\Delta^2} \\
&+ \sum_{cyc}\left(\frac{\gamma\alpha(\beta+\gamma-\alpha)\,v}{\Delta} + \frac{\beta\gamma(\gamma+\alpha-\beta)\,u}{\Delta}\right) \\
&= \sum_{cyc} \gamma uv + \frac{\alpha\beta\gamma}{\Delta^2}\sum_{cyc}\left(\gamma^2 - (\alpha-\beta)^2\right) + \sum_{cyc} \frac{u\beta\gamma(\gamma+\alpha-\beta+\alpha+\beta-\gamma)}{\Delta} \\
&= \sum_{cyc} \gamma uv + \frac{\alpha\beta\gamma}{\Delta} + 2\alpha\beta\gamma(u+v+w) = \sum_{cyc} \gamma uv + \frac{\alpha\beta\gamma}{\Delta} \\
&\leq \frac{\alpha\beta\gamma}{\Delta},
\end{aligned}$$

by **Lemma 1.**

Equality occurs when $x = \frac{\alpha(\beta+\gamma-\alpha)}{\Delta},\ y = \frac{\beta(\gamma+\alpha-\beta)}{\Delta}, z = \frac{\gamma(\alpha+\beta-\gamma)}{\Delta}$, and is implied by $u = v = w = 0$ in (A).

Thus equality occurs if and only if

$$x : y : z = (\beta + \gamma - \alpha)\, a : \beta\,(\gamma + \alpha - \beta) : \gamma\,(\alpha + \beta - \gamma)\,.$$

ii. Whenever $x + y + z = 0$ inequality (C) becomes (A).

2 Geometric Variations

Notations

1. Let K be the area of triangle ABC
2. Let R, r, s be the circumradius, inradius, and semiperimeter, respectively, of triangle ABC
3. $BC = a, CA = b, AB = c$
4. For an arbitrary interior point P, of triangle ABC, let the distance from P to vertex $X \in \{A, B, C\}$ be $R_X\,(P)$ or shortly R_X
5. Let the distance from P to the side $x \in \{a, b, c\}$ be $d_x\,(P)$ or shortly d_x
6. Let A_p, B_p, C_p be the feet of the perpendiculars from P onto sides BC, CA, AB
7. We will call the triangle $A_pB_pC_p$ the *Pedal Triangle* associated with P
8. $a_p = B_pC_p,\ b_p = C_pA_p,\ c_p = A_pB_p$.

Because R_a, R_b, R_c are diameters of the circumcircles of $PC_pAB_p, PA_pBC_p, PB_pCA_p$ then, by the Law of Sines, $\sin\alpha = \dfrac{a}{2R}, \sin\beta = \dfrac{b}{2R}, \sin\gamma = \dfrac{c}{2R}$ and

$$a_p = R_a \sin\alpha = \frac{aR_a}{2R},\ b_p = R_b \sin\beta = \frac{bR_b}{2R}, c_p = R_c \sin\gamma = \frac{cR_c}{2R}. \qquad (F1)$$

Let $K_a = K_{CPB},\ K_b = K_{APC},\ K_c = K_{BPA}$ and (p_a, p_b, p_c) be the baricentric coordinates of P, that is $p_a + p_b + p_c = 1$ and $p_a, p_b, p_c \geq 0$.

Then $p_a : p_b : p_c = K_a : K_b : K_c$, i.e. $p_a = \dfrac{K_a}{K}, p_b = \dfrac{K_b}{K}, p_c = \dfrac{K_c}{K}$. From $K_a + K_b + K_c = K$ it follows that

$$ad_a + bd_b + cd_c = 2F. \qquad (I)$$

Furthermore, $K_x = \dfrac{xd_x}{2},\ K = \dfrac{xh_x}{2}\ ,\ x \in \{a, b, c\}$ and $4KR = abc$, hence

$$d_a = \frac{2p_aK}{a} = \frac{p_abc}{2R}\ ,\ d_b = \frac{2p_bK}{b} = \frac{p_bca}{2R}, d_c = \frac{2p_cK}{c} = \frac{p_cab}{2R}. \qquad (F2)$$

Note that if a, b, and c are the sidelengths of a triangle, then

$$\Delta(a,b,c) > 0 \quad \text{and} \quad \Delta\left(a^2,b^2,c^2\right) = 16K^2.$$

Applications

Problem 1. Let P be an interior point of triangle ABC. Let a_p, b_p, c_p be the sides of the pedal triangle associated with P. Find the minimum value of

$$a_p^2 + b_p^2 + c_p^2.$$

Solution. Let (p_a, p_b, p_c) be the baricentric coordinates of P, that is $p_a, p_b, p_c \geq 0,\ p_a + p_b + p_c = 1$. Because $\angle B_pPC_p = 180^\circ - A$, by the Law of Cosines,$a_p^2 = d_b^2 + d_c^2 + 2d_bd_c\cos A$ and $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$. From (F2), $d_b = \dfrac{2p_bK}{b},\ d_c = \dfrac{2p_cK}{c}$, and $\cos A = \dfrac{b^2+c^2-a^2}{2bc}$ and

$$\begin{aligned}
a_p^2 &= \frac{4p_b^2K^2}{b^2} + \frac{4p_c^2K^2}{c^2} + \frac{4p_bp_cK^2}{bc}\cdot\frac{b^2+c^2-a^2}{bc} \\
&= \frac{4K^2}{b^2c^2}\left(p_b^2c^2 + p_c^2b^2 + p_bp_c\left(b^2+c^2-a^2\right)\right) \\
&= \frac{4K^2}{b^2c^2}\left(p_b\left(1-p_c-p_a\right)c^2 + p_c\left(1-p_a-p_b\right)b^2 + p_bp_c\left(b^2+c^2-a^2\right)\right) \\
&= \frac{4K^2}{b^2c^2}\left(p_bc^2 + p_cb^2 - p_bp_ca^2 - p_cp_ab^2 - p_ap_bc^2\right).
\end{aligned}$$

Then

$$\begin{aligned}
\sum_{cyc} a_p^2 &= \frac{4K^2}{a^2b^2c^2}\sum_{cyc}\left(p_bc^2a^2 + p_ca^2b^2 - p_bp_ca^4 - p_cp_aa^2b^2 - p_ap_ba^2c^2\right) \\
&= \frac{4K^2}{a^2b^2c^2}\left(2\sum_{cyc}p_ab^2c^2 - \left(a^2+b^2+c^2\right)\sum_{cyc}a^2p_bp_c\right).
\end{aligned}$$

Let $x = p_a, y = p_b, z = p_c,\ \alpha = \dfrac{a^2}{a^2+b^2+c^2}, \beta = \dfrac{b^2}{a^2+b^2+c^2}, \gamma = \dfrac{c^2}{a^2+b^2+c^2}$. Then $x+y+z = \alpha+\beta+\gamma = 1$. In addition, $\Delta(\alpha,\beta,\gamma) = \dfrac{16K^2}{\left(a^2+b^2+c^2\right)^2} > 0$ and

$$\min_P\left(a_p^2+b_p^2+c_p^2\right) = \frac{4K^2\left(a^2+b^2+c^2\right)^2}{a^2b^2c^2}\min_{x,y,z}\left(2\sum_{cyc}x\beta\gamma - \sum_{cyc}\alpha yz\right).$$

Recalling inequality (B) we get $\min\limits_{x,y,z}\left(2\sum\limits_{cyc}x\beta\gamma - \sum\limits_{cyc}\alpha yz\right) = 3\alpha\beta\gamma$. Then

$$\min_P \left(a_p^2 + b_p^2 + c_p^2\right) = \frac{4K^2\left(a^2+b^2+c^2\right)^2}{a^2b^2c^2} \cdot 3\alpha\beta\gamma = \frac{12K^2}{a^2+b^2+c^2}, \text{ or}$$

$$a_p^2 + b_p^2 + c_p^2 \geq \frac{12K^2}{a^2+b^2+c^2} \qquad (PT)$$

which is called the Pedal Triangle inequality. Equality occurs if and only if

$$p_a = \frac{a^2}{a^2+b^2+c^2}, p_b = \frac{b^2}{a^2+b^2+c^2}, p_c = \frac{c^2}{a^2+b^2+c^2}$$

or P is the Lemoine point of the given triangle.

Remark 1. Because $a_p = \dfrac{aR_a}{2R}$, $b_p = \dfrac{bR_b}{2R}, c_p = \dfrac{cR_c}{2R}$ and $aR_a + bR_b + cR_c \geq 4K$, we obtain

$$a_p^2 + b_p^2 + c_p^2 \geq \frac{(a_p+b_p+c_p)^2}{3} = \frac{1}{3}\left(\frac{aR_a}{2R} + \frac{bR_b}{2R} + \frac{cR_c}{2R}\right)^2$$
$$= \frac{(aR_a+bR_b+cR_c)^2}{12R^2} \geq \frac{4K^2}{3R^2}.$$

Since $\dfrac{12K^2}{a^2+b^2+c^2} \geq \dfrac{4K^2}{3R^2} \iff a^2+b^2+c^2 \leq 9R^2$, the inequality (PT) is sharper than $a_p^2 + b_p^2 + c_p^2 \geq \dfrac{4K^2}{3R^2}$.

Remark 2. If we let $a_p = \dfrac{aR_a}{2R}, b_p = \dfrac{bR_b}{2R}, c_p = \dfrac{cR_c}{2R}$, from (PT) it follows that

$$\sum_{cyc} a^2R_a^2 \geq \frac{3a^2b^2c^2}{a^2+b^2+c^2}. \qquad (RP)$$

Problem 2. Let a, b, and c be the sidelengths of a triangle ABC. Find the maximum value of

$$d_a(P)\,d_b(P) + d_b(P)d_c(P) + d_c(P)d_a(P),$$

where P is an arbitrary interior point of ABC.

Solution. Because $ad_a + bd_b + cd_c = 2K$, then by replacing (x, y, z) and (α, β, γ) in inequality (C) with (ad_a, bd_b, cd_c) and (a, b, c), respectively, we obtain

$$abc\,(d_bd_c + d_cd_a + d_ad_c) \leq \frac{abc\,(ad_a+bd_b+cd_c)^2}{\Delta(a,b,c)}$$

which is equivalent to

$$d_ad_b + d_bd_c + d_cd_a \leq \frac{4K^2}{\Delta(a,b,c)}. \qquad (DP)$$

Equality occurs if and only if $p_a : p_b : p_c = a(s-a) : b(s-b) : c(s-c)$ or $d_a : d_b : d_c = (s-a) : (s-b) : (s-c)$. This is equivalent to

$$d_a = \frac{2(s-a)}{\Delta(a,b,c)}, d_b = \frac{2(s-b)}{\Delta(a,b,c)}, d_c = \frac{2(s-c)}{\Delta(a,b,c)}.$$

Thus

$$\min\left(d_a(P)\,d_b(P) + d_b(P)d_c(P) + d_c(P)d_a(P)\right) = \frac{4K^2}{\Delta(a,b,c)}.$$

3 Algebraic Variations of inequalities (A),(B),(C).

Variation 1. Because $\Delta(b+c, c+a, a+b) = 4(ab+bc+ca) > 0$, by replacing (α, β, γ) in (C) with $(b+c, c+a, a+b)$ we obtain the inequality

$$ax(y+z) + by(z+x) + cz(x+y) \le \frac{(x+y+z)^2}{4} \cdot \frac{(a+b)(b+c)(c+a)}{ab+bc+ca} \qquad (D)$$

which holds for all positive real numbers a, b, c and all real numbers x, y, z. Equality occurs if and only if $x : y : z = a(b+c) : b(c+a) : c(a+b)$.

Because $ax(y+z) + by(z+x) + cz(x+y) = xy(a+b) + yz(b+c) + zx(c+a)$, inequality (D) can be rewritten in the form

$$\frac{xy}{(b+c)(c+a)} + \frac{yz}{(c+a)(a+b)} + \frac{zx}{(a+b)(b+c)} \le \frac{(x+y+z)^2}{4(ab+bc+ca)} \qquad (E)$$

and, by replacing (x, y, z) in (E) with $(x(b+c), y(c+a), z(a+b))$,we obtain

$$xy + yz + zx \le \frac{(x(b+c) + y(c+a) + z(a+b))^2}{4(ab+bc+ca)}. \qquad (F)$$

Variation 2. Inequality (F) written as

$$(a(y+z) + b(z+x) + c(x+y))^2 \ge 4(xy+yz+zx)(ab+bc+ca) \qquad (F1)$$

is identical to the inequality from [4].

Assume that $x, y, z \ge 0$ in (F1). Then

$$x(b+c) + y(c+a) + z(a+b) \ge 2\sqrt{(ab+bc+ca)(xy+yz+zx)} \qquad (F2)$$

and

$$\frac{x(b+c)}{y+z} + \frac{y(c+a)}{z+x} + \frac{z(a+b)}{x+y} \ge \sqrt{3(ab+bc+ca)}, \qquad (F3)$$

(can be obtained by replacing (x, y, z) in (F2) with $\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)$ and using the inequality $\sum_{cyc} \frac{yz}{(z+x)(x+y)} \geq \frac{3}{4} \iff 9xyz \leq (x+y+z)(xy+yz+zx)$) were discussed in [2] and [3] as strong tools for proving several hard inequalities.

Adding $ax+by+cz$ to both sides of (F2) we obtain

$$(a+b+c)(x+y+z) \geq ax+by+cz+2\sqrt{(ab+bc+ca)(xy+yz+zx)}. \quad (F4)$$

Inequality (F4) is the homogeneous form of an inequality from the **2001 Ukrainian Mathematics Olympiad,** which also appeared in [1] as problem 6.

We conclude this article with a problem.
Problem 3. For positive real numbers $x_1, x_2, x_3, x_4, x_5, x_6$ prove that

$$\sum_{cyc}^{6} x_1x_2x_3x_4 \leq \frac{(x_1+x_2+x_3+x_4+x_5+x_6)^4}{6^3}.$$

Solution. Because

$$\sum_{cyc}^{6} x_1x_2x_3x_4 = x_1x_3 \cdot x_2(x_4+x_6) + x_3x_5 \cdot x_4(x_6+x_2) + x_5x_1 \cdot x_6(x_2+x_4),$$

applying inequality (C) for $a = x_1x_3, b = x_3x_5, c = x_5x_1$ and $x = x_2, y = x_4, z = x_6$ we obtain:

$$\sum_{cyc}^{6} x_1x_2x_3x_4 \leq \frac{(x_2+x_4+x_6)^2}{4} \cdot \frac{(x_1x_3+x_3x_5)(x_3x_5+x_5x_1)(x_5x_1+x_1x_3)}{x_1x_3x_5(x_1+x_3+x_5)}$$
$$= \frac{(x_2+x_4+x_6)^2}{4} \cdot \frac{(x_1+x_3)(x_3+x_5)(x_5+x_1)}{x_1+x_3+x_5}.$$

By the AM-GM inequality,

$$\frac{(x_1+x_3)(x_3+x_5)(x_5+x_1)}{(x_1+x_3+x_5)^3} \leq \left(\frac{\frac{x_1+x_3}{x_1+x_3+x_5}+\frac{x_3+x_5}{x_1+x_3+x_5}+\frac{x_5+x_1}{x_1+x_3+x_5}}{3}\right)^3$$
$$= \left(\frac{2}{3}\right)^3 = \frac{8}{27},$$

hence

$$\frac{(x_1+x_3)(x_3+x_5)(x_5+x_1)}{x_1+x_3+x_5} \leq \frac{8}{27}(x_1+x_3+x_5)^2$$

and

$$
\begin{aligned}
\sum_{cyc}^{6} x_1x_2x_3x_4 &\le \frac{(x_2+x_4+x_6)^2}{4} \cdot \frac{8}{27}(x_1+x_3+x_5)^2 \\
&= \frac{2}{27}(x_2+x_4+x_6)^2 \cdot (x_1+x_3+x_5)^2 \le \frac{2}{27} \cdot \left(\frac{x_1+x_2+\cdots+x_6}{2}\right)^4 \\
&= \frac{(x_1+x_2+x_3+x_4+x_5+x_6)^4}{6^3}.
\end{aligned}
$$

References.

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